GRADIENT ESTIMATE FOR EIGENFORMS OF HODGE LAPLACIAN

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Abstract. In this paper, we derive a gradient estimate for the linear combinations of eigenforms of the Hodge Laplacian on a closed manifold. The estimate is given in terms of the dimension, volume, diameter and curvature bound of the manifold. As an application, we obtain directly a sharp estimate for the heat kernel of the Hodge Laplacian.

1. Introduction

Let (M^n, g) be a compact oriented Riemannian manifold without boundary. The Hodge Laplacian $\Delta : A^p(M) \to A^p(M)$, acting on the space of smooth p-forms $A^p(M)$ on M, is defined by

$$\Delta = -d\delta - \delta d.$$

Here, as usual, d is the exterior differential operator and δ the adjoint of d with respect to the L^2 inner product on $A^p(M)$. We denote the eigenvalues of Δ by $\{0 \leq \lambda_1 \leq \ldots \lambda_k \leq \ldots\}$ with a corresponding orthonormal basis of eigenforms $\{\phi_i\}_{i=1}^{\infty}$. We have the following estimate concerning the eigenforms.

Theorem 1.1. *Let (M^n, g) be a closed manifold with curvature bound $|Rm| \leq K$. Then for any $b_i \in \mathbb{R}$ with $\sum_{i=1}^k b_i^2 \leq 1$, the form $\omega = \sum_{i=1}^k b_i \phi_i$ satisfies the estimate*

$$|\nabla \omega|^2 + (\lambda_k + 1)\, |\omega|^2 \leq c\, (\lambda_k + 1)^{\frac{n}{2}+1},$$

where $c = c(n, V, d, K)$ is an explicit constant depending on the dimension n, volume V, diameter d and the curvature bound K.

We would like to emphasize that the estimate is valid for all finite linear combinations of the eigenforms, and it does not involve any covariant derivatives of the curvature tensor. Also, the exponent $\frac{n}{2} + 1$ in λ_k is sharp. This sharp exponent in turn leads to another one in k for the lower bound of the eigenvalue $\lambda_k \geq c\, k^{-\frac{2}{n}}$ for all $k > b_p$, the Betti number of the $p-$th cohomology of M.

Our estimates can then be applied to analyze the heat kernel of Δ. Combining with a result of Rumin [9], one has the following Sobolev inequality for $p-$forms.

Theorem 1.2. *For an explicit constant $c = c(n, V, d, K)$,*

$$\left(\int_M |\omega - P(\omega)|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq c \int_M \{|d\omega|^2 + |\delta\omega|^2\}$$

for all smooth $p-$form ω on M, where $P(\omega)$ denotes the projection of ω on to the space of harmonic $p-$forms.

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Another consequence is the following Hessian estimate for the eigenfunctions on M.

Corollary 1.3. *Let (M^n, g) be a closed manifold with curvature bound $|Rm| \leq K$. Let $\phi_1, \phi_2, \cdots, \phi_k$ be orthonormal eigenfunctions of the scalar Laplacian with corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then there exists a constant $c(K, d, V, n)$ such that*

$$|\nabla d\, f| \leq c\, \lambda_k^{\frac{n+4}{4}},$$

where $f = \sum_{i=1}^k b_i\, \phi_i$ and $\sum_{i=1}^k b_i^2 = 1$.

Let us point out that the analysis of the Laplacian on a compact manifold is a classical subject. Numerous contributions have been made by various authors. While the results here are mostly known, we do hope our seemingly more direct treatment is of certain expository value.

As well-known, the gradient estimate method was successfully employed by Yau [11] to study harmonic functions on complete manifolds. The method was further developed by Li [5], and Li and Yau [7] to study eigenfunctions and eigenvalues. In particular, they have obtained a lower bound for the first non-zero eigenvalue of the scalar Laplacian in terms of the lower bound of Ricci curvature and the diameter of the manifold.

Our current work is very much motivated by and follows the ideas in a famous paper of Li [6], where he has obtained a lower bounds for higher eigenvalues of the Hodge Laplacian. This is achieved through an estimate of the linear combinations of the eigenforms. The estimate involves the curvature operator lower bound and the Sobolev constant of the manifold, but not the curvature upper bound. However, the estimate there seems insufficient to provide a sharp exponent for the eigenvalue lower bounds alluded above. We would also like to point out that both E. Aubry's PhD thesis and the paper [1] by W. Ballmann, J. Brüning and G. Carron have already developed a gradient estimate for individual eigenforms.

The case of scalar Laplacian deserves special attention as it is of more common concern. So we will treat it separately in section 2. The result is a bit stronger in the sense it only involves the Ricci curvature lower bound in all the estimates. The approach is also more straightforward as it only relies on a direct application of the maximum principle.

The case of general Hodge Laplacian is handled in section 3. The proof now involves an iteration scheme as in [6].

Finally, we mention that the results here can be extended to the case of compact manifolds with boundary. For the ease of exposition, we omit the details here.

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2. Analysis of scalar Laplacian

In this section, we will derive a variant version of the well-known gradient estimates of Li-Yau[7] concerning the eigenfunctions. As an application, we give direct

proofs to some well-known results including a lower bound of the high eigenvalue, the existence of heat kernel and its long time decay estimate.

Let (M^n, g) be a closed Riemannian manifold with diameter d, volume V, and Ricci curvature lower bound $-(n-1)K$, where $K \geq 0$ is a constant. Denote the eigenvalues of the Laplacian Δ by $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$ with the corresponding eigenfunction ϕ_i, $i = 0, 1, 2, \cdots$, satisfying

$$\Delta\phi_i = -\lambda_i\phi_i, \quad \int_M \phi_i\,\phi_j = \delta_{ij}.$$

For a given constant c, consider the function

$$Q(x) = |\nabla\phi|^2 + c\,\phi^2,$$

where $\phi = \sum_{i=1}^k b_i\phi_i$ with $b_i \in \mathbb{R}$ and $\sum_{i=1}^k b_i^2 = 1$. Obviously, the maximum value of $Q(x)$ over M is a function of $b_1, \cdots, b_k$. This function in turn achieves its maximum at some point $a_1, \cdots, a_k$. Let $u = \sum_{i=1}^k a_i\phi_i$.

Lemma 2.1.

$$|\nabla u|^2 + A\,u^2 \leq A \max_M u^2,$$

where $A = \lambda_k + (n-1)K$.

Proof. Define

$$F(b_1, \ldots, b_k, x, \lambda) = Q(x) - \lambda\,(\sum_{i=1}^k b_i^2 - 1).$$

Then, subject to the constraint $\sum_{i=1}^k b_i^2 = 1$, F achieves its maximum value at some point $(a_1, \cdots, a_k, x_0, \alpha)$. We now show

$$|\nabla u|^2(x_0) + c\,u^2(x_0) \leq c \max_M u^2$$

for $c > \lambda_k + (n-1)K$.

At the point $(a_1, \cdots, a_k, x_0, \alpha)$, F satisfies

$$\begin{cases} \nabla F(a_1, \cdots, a_k, x_0, \alpha) = 0 \\ \Delta F(a_1, \cdots, a_k, x_0, \alpha) \leq 0 \\ \frac{\partial F}{\partial b_i} = 0 \\ \sum_{i=1}^k a_i^2 = 1. \end{cases} \tag{2.1}$$

From the third equation of (2.1), we have

$$\sum_{j=1}^k (2a_j\langle\nabla\phi_i, \nabla\phi_j\rangle + 2ca_j\langle\phi_i, \phi_j\rangle) - 2\alpha a_i = 0.$$

After multiplying by a_i and summing over i, one sees that

$$\alpha = Q(u, x_0) = |\nabla u|^2(x_0) + c\,u^2(x_0).$$

Suppose now that

$$|\nabla u|^2(x_0) + cu^2(x_0) > c \max_M u^2.$$

Then

$$\nabla u(x_0) \neq 0$$

and one can choose an orthonormal frame $\{e_1, \ldots, e_n\}$ at x_0 so that

$$\nabla u(x_0) = u_1(x_0) e_1.$$

Now the first equation of (2.1), $\nabla F(a_1, \ldots, a_k, x_0, \alpha) = 0$, becomes

$$2u_1 u_{1i} + 2cuu_i = 0$$

for $i = 1, \ldots, n$. This in particular implies

$$|\nabla\nabla u|^2 \geq u_{11}^2 = c^2\, u^2. \tag{2.2}$$

On the other hand, at the maximum point $(a_1, \cdots, a_k, x_0, \alpha)$,

$$\Delta F(a_1, \ldots, a_k, x_0, \alpha) \leq 0$$

or

$$\Delta |\nabla u|^2 + c\Delta u^2 \leq 0. \tag{2.3}$$

By the Bochner formula, it becomes

$$|\nabla\nabla u|^2 + \langle \nabla \Delta u, \nabla u\rangle + \langle Ric(\nabla u, \nabla u)\rangle + cu\Delta u + c|\nabla u|^2 \leq 0.$$

In view of (2.2) and the lower bound of Ricci curvature, the above inequality reduces to

$$c^2 u^2 + \langle \nabla \Delta u, \nabla u\rangle - (n-1)K|\nabla u|^2 + cu\Delta u + c|\nabla u|^2 \leq 0.$$

Note that

$$\Delta u = -\sum_{i=1}^k \lambda_i\, a_i\, \phi_i.$$

Therefore,

$$\begin{aligned}
0 \quad &\geq \quad c^2\, u^2 + (c - (n-1)K)|\nabla u|^2 - \sum_{i,j=1}^k \lambda_i\, a_i\, a_j\, \langle \nabla\phi_i, \nabla\phi_j\rangle - \sum_{i,j=1}^k c\, \lambda_i\, a_i\, a_j\, \phi_i, \phi_j \\
&\geq \quad c^2\, u^2 + (c - (n-1)K)|\nabla u|^2 - \sum_{i=1}^k \lambda_i\, a_i \sum_{j=1}^k (a_j \langle \nabla\phi_i, \nabla\phi_j\rangle + c\, a_j\, \phi_i\, \phi_j) \\
&\geq \quad c^2\, u^2 + (c - (n-1)K)|\nabla u|^2 - \sum_{i=1}^k \alpha\, \lambda_i\, a_i^2 \\
&\geq \quad c^2\, u^2 + (c - (n-1)K)|\nabla u|^2 - \alpha\, \lambda_k \\
&\geq \quad c^2\, u^2 + (c - (n-1)K)|\nabla u|^2 - \lambda_k\, (|\nabla u|^2 + cu^2) \\
&\geq \quad c\, (c - \lambda_k)\, u^2 + (c - (n-1)K - \lambda_k)|\nabla u|^2.
\end{aligned}$$

This is an obvious contradiction if $c > (n-1)K + \lambda_k$. In other words,

$$|\nabla u|^2(x_0) + c\, u^2(x_0) \leq c \max_M u^2$$

for all $c > (n-1)K + \lambda_k$. The lemma follows by letting c approach $\lambda_k + (n-1)K$. $\square$

As a consequence, we obtain a quick proof to the following well-known facts.

Theorem 2.2. *There exists a constant $c(K,d,V,n)$ such that*
(1)
$$|\nabla\phi|^2 \le c\lambda_k^{\frac{n+2}{2}}, \quad \phi^2 \le c\lambda_k^{\frac{n}{2}}.$$
In particular,
$$|\nabla\phi_k| \le c\lambda_k^{\frac{n+2}{4}}, \quad |\phi_k| \le c\lambda_k^{\frac{n}{4}}.$$
(2) For all $k \ge 1$,
$$\lambda_k \ge c^{-1}\, k^{\frac{2}{n}}.$$
(3) The function $H(x,y,t)$ given by
$$H(x,y,t) = \frac{1}{V} + \sum_{k=1}^{\infty} e^{-\lambda_k t}\,\phi_k(x)\,\phi_k(y)$$
is a heat kernel of M. Moreover,
$$|H(x,y,t) - \frac{1}{V}| \le c\,t^{-\frac{n}{2}}$$
for all $t > 0$.
(4) The following Sobolev inequality holds.
$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le c\int_M |\nabla f|^2$$
for all smooth function f on M with $\int_M f = 0$.

Proof. (1) Let u be the function considered in the preceding lemma. Then we need only to prove the estimate for u. Choose point p such that
$$u^2(p) = \max_M u^2.$$
For $r > 0$ and $x \in B_p(\frac{r}{\sqrt{\lambda_k + (n-1)K}})$, using lemma 2.1, we conclude
$$\begin{aligned} u^2(p) - u^2(x) &\le \max_{y\in M} 2|u|(y)\,|\nabla u|(y)\,d(x,p) \\ &\le 2\,u^2(p)\sqrt{\lambda_k + (n-1)K}\,\frac{r}{\sqrt{\lambda_k + (n-1)K}} \\ &\le 2r\,u^2(p). \end{aligned}$$
Therefore,
$$u^2(x) \ge (1-2r)\,u^2(p)$$
on $B_p(\frac{r}{\sqrt{\lambda_k + (n-1)K}})$. Integrating with respect to x over the ball yields
$$1 = ||u||^2_{L^2(M)} \ge (1-2r)\,u^2(p)\,\frac{V_p(\frac{r}{\sqrt{\lambda_k + (n-1)K}})}{V_p(d)}\,V_p(d).$$
Choose r such that
$$4r < 1 \quad \text{and} \quad \frac{r}{\lambda_1 + \sqrt{(n-1)K}} < d.$$
Then by the Bishop volume comparison theorem we have

$$1 \geq (1-2r)\, u^2(p)\, \frac{c(K,d,V,n)}{(\lambda_k+(n-1)K)^{\frac{n}{2}}}.$$

In other words,

$$u^2(x) \leq u^2(p) \leq c(K,d,V,n)\, \lambda_k^{\frac{n}{2}},$$

where we have used the fact that $\lambda_k \geq \lambda_1 \geq c$ by [7]. Using lemma 2.1 again, we also conclude

$$|\nabla u|^2(x) \leq (\lambda_k+(n-1)K)\, u^2(p) \leq c(K,d,V,n)\, \lambda_k^{\frac{n+2}{2}}.$$

(2) For each $x \in M$, there exists an orthogonal matrix $(a_{ij})_{k\times k}$ such that

$$\nabla \psi_j(x) = 0$$

for $j = n+1, \cdots, k$, where $\psi_j = \sum_{i=1}^k a_{ij}\, \phi_i$.

From (1), it follows that

$$\begin{aligned} \sum_{i=1}^k |\nabla \phi_i|^2(x) &= \sum_{j=1}^n |\nabla \psi_j|^2(x) \\ &\leq n \max_j |\nabla \psi_j|^2 \\ &\leq c_1\, \lambda_k^{\frac{n+2}{2}}. \end{aligned}$$

Integrating the inequality with respect to x, we conclude

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq c_2\, \lambda_k^{\frac{n+2}{2}}.$$

By an elementary induction argument, the inequality implies

$$\lambda_k \geq c_3\, k^{2/n}$$

for all $k \geq 1$, where $c_3 = \min\{\lambda_1, (\frac{1}{c_2}\frac{n}{n+2})^{\frac{n}{2}}\}$.

(3) In view of (1) and (2), it is straightforward to check the infinite series

$$\frac{1}{V} + \sum_{k=1}^{\infty} e^{-\lambda_k t}\, \phi_k(x)\, \phi_k(y)$$

converges uniformly in the C^1 sense for $x, y \in M$ and $t \geq c$ for any $c > 0$. It is then easy to verify the limit is a heat kernel of M.

Since

$$c^{-1}\, k^{\frac{2}{n}} \leq \lambda_k \leq c\, k^{\frac{2}{n}},$$

one sees by (1) that

$$\begin{aligned} |H(x,y,t) - \frac{1}{V}| &\leq \sum_{k=1}^{\infty} e^{-\lambda_k t} |\phi_k|(x)\, |\phi_k|(y) \\ &\leq \sum_{k=1}^{\infty} c\, \lambda_k^{\frac{n}{2}}\, e^{-\lambda_k t} \\ &\leq c\, t^{-\frac{n}{2}} \int_0^{\infty} s^{\frac{n}{2}}\, e^{-s}\, ds \\ &\leq c\, t^{-\frac{n}{2}}. \end{aligned}$$

(4) follows from (3) by a result of Varopoulos [10]. □

We remark that both (2) and (3) were first proved by Cheng and Li [3] using the Sobolev inequality. Historically, the Sobolev inequality on manifolds was derived from the isoperimetric inequalities, which were established by Yau [12] and Croke [2].

3. Gradient estimate for eigenforms

Using the well-known Bochner-Weitzenbock formula, one can directly apply the proof in the previous section to the Hodge Laplacian acting on the smooth $p-$forms on M. However, the resulting estimates depend also on the bounds of the covariant derivative of the curvature tensor of M. It turns out this dependency is superfluous by adopting a different argument as demonstrated by E. Aubry in his PhD thesis and also by W. Ballmann, J. Brüning and G. Carron in [1]. In the following, we present a slightly refined version of their argument to suit our purpose.

We will use the moving frame notations. So for a $p-$form ω on M, under an orthonormal coframe $\{\omega_1, \cdots, \omega_n\}$, $\omega = a_{i_1\cdots i_p}\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}$.

The Bochner-Weitzenbock formula says

$$\Delta\omega = \Delta_B\omega - E(\omega),$$

where

$$\Delta_B\omega = a_{i_1\cdots i_p,jj}\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}$$

is the Bochner Laplacian and

$$E(\omega) = R_{k_\beta i_\beta j_\alpha i_\alpha}\, a_{i_1\ldots k_\beta \ldots i_p}\, \omega_{i_p} \wedge \cdots \wedge \omega_{j_\alpha} \wedge \cdots \wedge \omega_{i_1}$$

with R_{ijkl} being the curvature tensor of M. Now,

$$\Delta_B(\nabla\omega) = a_{i_1\ldots i_p,ijj}\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \otimes \omega_i$$

and

$$\nabla\Delta\omega = a_{i_1\ldots i_p,jji}\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \otimes \omega_i - \nabla(E(\omega)).$$

Hence

$$\Delta_B(\nabla\omega) - \nabla\Delta\omega = a_{I,ijj}\omega_I \otimes \omega_i - a_{I,jji}\omega_I \otimes \omega_i + \nabla(E(\omega)).$$

By the Ricci identity, we have

$$a_{I,ijj} - a_{I,jij} = (R_{j_\alpha i_\alpha ij}a_{i_1\ldots j_\alpha \ldots i_p})_{,j}$$

and

$$a_{I,jij} - a_{I,jji} = R_{j_\alpha i_\alpha ij}a_{i_1\ldots j_\alpha\ldots i_p,j} + R_{ljij}a_{i_1\ldots i_p,l}.$$

Thus we have the commutation formula

$$\begin{aligned} \Delta_B(\nabla\omega) - \nabla\Delta\omega \;=\;& R_{li}a_{i_1\ldots i_p,l}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \otimes \omega_i \\ +\;& R_{j_\alpha i_\alpha ij}a_{i_1\ldots j_\alpha\ldots i_p,j}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \otimes \omega_i \\ +\;& \left(R_{j_\alpha i_\alpha ij}\, a_{i_1\ldots j_\alpha\ldots i_p}\right)_{,j}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \otimes \omega_i \\ +\;& \nabla(E(\omega)). \end{aligned} \tag{3.1}$$

Finally, we conclude

$$\begin{aligned}\langle \Delta_B(\nabla\omega) - \nabla\Delta\omega, \nabla\omega\rangle &= R_{li} a_{i_1\dots i_p,l} a_{i_1\dots i_p,i} \\ &+ R_{j_\alpha i_\alpha ij} a_{i_1\dots j_\alpha \dots i_p,j} a_{i_1\dots i_\alpha\dots i_p,i} \\ &+ \left(R_{j_\alpha i_\alpha ij}\, a_{i_1\dots j_\alpha\dots i_p}\right)_{,j} a_{i_1\dots i_\alpha\dots i_p,i} \\ &+ \langle \nabla(E(\omega)), \nabla\omega\rangle \end{aligned} \tag{3.2}$$

Note that these formulas and the following lemma have more or less been derived by Le Couturier and G. Robert in [4].

We now consider the function $f = |\nabla\omega|^2 + A\,|\omega|^2$, where $A \geq 1$ is a fixed constant.

Lemma 3.1. *Let (M^n, g) be a closed Riemannian manifold with curvature operator $|Rm| \leq K$. Then for $k \geq 1$,*

$$\int_M f^{k-1}\Delta f \geq 2\int_M \left(\langle \nabla\Delta\omega, \nabla\omega\rangle + A\,\langle \Delta\omega, \omega\rangle\right) f^{k-1} - c\,k^2 \int_M f^k,$$

where $c = 2nK(K+2) + 18\,K^2$.

Proof. Direct calculation gives

$$\begin{aligned}\Delta f &= \Delta\left(|\nabla\omega|^2 + A\,|\omega|^2\right) \\ &= 2\langle \Delta_B(\nabla\omega), \nabla\omega\rangle + 2|\nabla\nabla\omega|^2 \\ &+ 2A\,|\nabla\omega|^2 + 2A\,\langle \Delta_B\omega, \omega\rangle \\ &= 2\langle \nabla\Delta\omega, \nabla\omega\rangle + 2A\,\langle \Delta\omega, \omega\rangle \\ &+ 2|\nabla\nabla\omega|^2 + 2A\,|\nabla\omega|^2 + 2A\,\langle E(\omega), \omega\rangle \\ &+ 2\,\langle \Delta_B(\nabla\omega) - \nabla\Delta\omega, \nabla\omega\rangle. \end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned}\int_M f^{k-1}\Delta f &= 2\int_M \left(\langle \nabla\Delta\omega, \nabla\omega\rangle + A\,\langle \Delta\omega, \omega\rangle\right) f^{k-1} \\ &+ 2\int_M \left(|\nabla\nabla\omega|^2 + A\,|\nabla\omega|^2\right)\, f^{k-1} \\ &+ 2A\int_M \langle E(\omega), \omega\rangle\, f^{k-1} \\ &+ 2\int_M \langle \Delta_B(\nabla\omega) - \nabla\Delta\omega, \nabla\omega\rangle\, f^{k-1}. \end{aligned} \tag{3.4}$$

Since $|Rm| \leq K$,

$$2A\int_M \langle E(\omega), \omega\rangle\, f^{k-1} \geq -2K\int_M f^k. \tag{3.5}$$

Using (3.2), we have

$$
\begin{aligned}
(3.6)\qquad & 2\int_M \langle \Delta_B(\nabla\omega) - \nabla\Delta\omega, \nabla\omega\rangle\, f^{k-1} \\
= \;& 2\int_M R_{li} a_{i_1\ldots i_p,l} a_{i_1\ldots i_p,i}\, f^{k-1} \\
+ \;& 2\int_M R_{j_\alpha i_\alpha ij} a_{i_1\ldots j_\alpha\ldots i_p,j} a_{i_1\ldots i_\alpha\ldots i_p,i}\, f^{k-1} \\
+ \;& 2\int_M \left(R_{j_\alpha i_\alpha ij}\, a_{i_1\ldots j_\alpha\ldots i_p}\right)_{,j} a_{i_1\ldots i_\alpha\ldots i_p,i}\, f^{k-1} \\
+ \;& 2\int_M \langle\nabla(E(\omega)), \nabla\omega\rangle f^{k-1}.
\end{aligned}
$$

The first and second term of (3.6) obviously satisfy

$$
(3.7)\qquad 2\int_M R_{li} a_{i_1\ldots i_p,l} a_{i_1\ldots i_p,i}\, f^{k-1} \geq -2(n-1)K \int_M f^k.
$$

and

$$
\begin{aligned}
(3.8)\qquad 2\int_M R_{j_\alpha i_\alpha ij} a_{i_1\ldots j_\alpha\ldots i_p,j} a_{i_1\ldots i_\alpha\ldots i_p,i}\, f^{k-1} \;&\geq\; -2K \int_M |\nabla\omega|^2 f^{k-1} \\
&\geq\; -2K \int_M f^k.
\end{aligned}
$$

For the third term of (3.6), after integration by parts, we have

$$
\begin{aligned}
(3.9)\qquad & 2\int_M \left(R_{j_\alpha i_\alpha ij}\, a_{i_1\ldots j_\alpha\ldots i_p}\right)_{,j} a_{i_1\ldots i_\alpha\ldots i_p,i}\, f^{k-1} \\
= \;& -2\int_M R_{j_\alpha i_\alpha ij}\, a_{i_1\ldots j_\alpha\ldots i_p} a_{i_1\ldots i_\alpha\ldots i_p,ij}\, f^{k-1} \\
- \;& 2(k-1)\int_M R_{j_\alpha i_\alpha ij}\, a_{i_1\ldots j_\alpha\ldots i_p} a_{i_1\ldots i_\alpha\ldots i_p,i}\, f^{k-2} f_j \\
\geq \;& -2K\int_M |\omega|\,|\nabla\nabla\omega|\, f^{k-1} \\
- \;& 2(k-1)K\int_M |\omega|\,|\nabla\omega|\, f^{k-2}\,|\nabla f| \\
\geq \;& -2K^2\int_M f^k - \frac{1}{2}\int_M |\nabla\nabla\omega|^2 f^{k-1} \\
- \;& 8\,k^2 K^2 \int_M f^k - \frac{1}{2}\int_M (|\nabla\nabla\omega|^2 + A\,|\nabla\omega|^2)\, f^{k-1},
\end{aligned}
$$

where we have used the fact that

$$
|\omega|\,|\nabla\omega| \leq f
$$

and

$$
\begin{aligned}
(3.10)\qquad |\nabla f| \;&\leq\; 2|\nabla\omega|\,|\nabla\nabla\omega| + 2A\,|\omega|\,|\nabla\omega| \\
&\leq\; 4k\,K\,(|\nabla\omega|^2 + A\,|\omega|^2) + \frac{1}{4k\,K}(|\nabla\nabla\omega|^2 + A\,|\nabla\omega|^2).
\end{aligned}
$$

Applying integration of parts to the last term of (3.6), we get

$$\begin{aligned}
(3.11)\qquad & 2\int_M \langle \nabla(E(\omega)), \nabla\omega\rangle f^{k-1} \\
\geq\ & -2\int_M \langle E(\omega), \Delta_B \omega\rangle f^{k-1} \\
-\ & 2(k-1)\int_M |E(\omega)|\,|\nabla\omega|\, f^{k-2}\,|\nabla f| \\
\geq\ & -2\sqrt{n}\,K\int_M |\omega|\,|\nabla\nabla\omega|\, f^{k-1} \\
-\ & 2(k-1)K\int_M |\omega|\,|\nabla\omega|\, f^{k-2}\,|\nabla f| \\
\geq\ & -2n\,K^2\int_M f^k - \frac{1}{2}\int_M |\nabla\nabla\omega|^2\, f^{k-1} \\
-\ & 8\,k^2\,K^2\int_M f^k - \frac{1}{2}\int_M (|\nabla\nabla\omega|^2 + A\,|\nabla\omega|^2)\, f^{k-1},
\end{aligned}$$

where we have used (3.10) in the last step.

Putting (3.7), (3.8), (3.9) and (3.11) into (3.6), we conclude

$$\begin{aligned}
(3.12)\qquad & 2\int_M \langle \Delta_B(\nabla\omega) - \nabla\Delta\omega, \nabla\omega\rangle\, f^{k-1} \\
\geq\ & -\left(2nK(K+1) + 18\,k^2\,K^2\right)\int_M f^k \\
-\ & 2\int_M (|\nabla\nabla\omega|^2 + A\,|\nabla\omega|^2)\, f^{k-1}.
\end{aligned}$$

Plugging (3.5) and (3.12) into (3.4), we arrived at

$$\begin{aligned}
\int_M f^{k-1}\Delta f \quad \geq\ & 2\int_M \left(\langle \nabla\Delta\omega, \nabla\omega\rangle + A\,\langle \Delta\omega, \omega\rangle\right)\, f^{k-1} \\
-\ & \left(2nK(K+2) + 18\,k^2\,K^2\right)\int_M f^k.
\end{aligned}$$

The lemma is proved. □

We now prove a gradient estimate concerning the linear combinations of eigenforms.

Theorem 3.2. *Let (M^n, g) be a closed manifold with curvature bound $|Rm| \leq K$. Let $\phi_1, \phi_2, \cdots, \phi_l$ be orthonormal eigenforms of the Hodge Laplacian Δ acting on the $p-$forms with corresponding eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$. Then for any $b_i \in \mathbb{R}$ with $\sum_{i=1}^l b_i^2 \leq 1$, the form $\omega = \sum_{i=1}^l b_i\phi_i$ satisfies the estimate*

$$|\nabla\omega|^2 + A\,|\omega|^2 \leq c\,(\lambda_l + K + 1)^{\frac{n}{2}+1},$$

where $A = \lambda_l + K + 1$, and $c = c(n, V, d, K)$ is a constant.

Proof. For each $k \geq 1$, let

$$I_k = \max \int_M f^{2k},$$

where $f = |\nabla\omega|^2 + A\,|\omega|^2$ and the maximum is taken over all $\omega = \sum_{i=1}^{l} b_i\,\phi_i$ with $b_i \in \mathbb{R}$ and $\sum_{i=1}^{l} b_i^2 \le 1$.

Note that for $\omega = \sum_{i=1}^{l} b_i\phi_i$ with $\sum_{i=1}^{l} b_i^2 \le 1$,

$$\Delta\omega = -\sum_{i=1}^{l} \lambda_i\, b_i\, \phi_i = -\lambda_l\, \sum_{i=1}^{l} a_i\phi_i,$$

where $a_i = \lambda_i\,\lambda_l^{-1}\, b_i$, $i = 1, \cdots, l$. Obviously,

$$\sum_{i=1}^{l} a_i^2 \le 1.$$

So if we denote $\eta = \sum_{i=1}^{l} a_i\,\phi_i$, then

$$\begin{aligned}
&\int_M (\langle\nabla\Delta\omega, \nabla\omega\rangle + A\,\langle\Delta\omega, \omega\rangle)\, f^{2k-1} \\
\ge\ & -\int_M \left(|\nabla\Delta\omega|^2 + A\,|\Delta\omega|^2\right)^{\frac{1}{2}}\, f^{2k-\frac{1}{2}} \\
\ge\ & -\lambda_l \left(\int_M (|\nabla\eta|^2 + A\,|\eta|^2)^{2k}\right)^{\frac{1}{4k}} \left(\int_M f^{2k}\right)^{\frac{4k-1}{4k}} \\
\ge\ & -\lambda_l\, I_k.
\end{aligned}$$

Combining with lemma 3.1, we have the estimate

$$\int_M f^{2k-1}\Delta f \ge -(2\,\lambda_l + c_1\,k^2)\, I_k, \tag{3.13}$$

where $c_1 = 8nK(K+2) + 72\,K^2$.

On the other hand

$$\int_M f^{2k-1}\Delta f = -\frac{2k-1}{k^2}\int_M |\nabla f^k|^2. \tag{3.14}$$

Applying the Sobolev inequality

$$\left(\int_M |u|^{2\beta}\right)^{\frac{1}{\beta}} \le C_s\,\left(\int_M |\nabla u|^2 + \int_M u^2\right),$$

where $\beta = \frac{n}{n-2}$, to $u = f^k$, we get

$$\left(\int_M f^{2k\beta}\right)^{\frac{1}{\beta}} \le C_s\,\left(\int_M |\nabla f^k|^2 + \int_M f^{2k}\right). \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we get

$$\left(\int_M f^{2k\beta}\right)^{\frac{1}{\beta}} \le C_s\, k\,(\lambda_l + c_1\,k^2)\, I_k$$

Since this is true for all ω, we may maximize the left hand side over ω and conclude

$$(I_{\beta k})^{\frac{1}{\beta k}} \le \left(C_s\, k\,(\lambda_l + c_1\,k^2)\right)^{\frac{1}{k}}\, (I_k)^{\frac{1}{k}}$$

for all $k \ge 1$.

Let $k=\beta^i$, $i=0,1,2,\cdots$ and iterate the preceding inequality. Then,

$$\begin{aligned}\lim_{i\to\infty}\left(I_{\beta^i}\right)^{\frac{1}{\beta^i}} &\le \prod_{i=0}^{\infty}\left(C_s\,\beta^i\,(\lambda_l+c_1\,\beta^{2i})\right)^{\frac{1}{\beta^i}}\;I_1\\ &\le c_2\,(\lambda_l+1)^{\frac{n}{2}}\,I_1,\end{aligned}$$

where $c_2=c_2(n,d,V,K)$ is a constant. In other words,

$$\begin{aligned}&\max_{\omega}\max_{x\in M}\left(|\nabla\omega|^2+A\,|\omega|^2\right)^2(x)\\ \le\;& c_2\,(\lambda_l+1)^{\frac{n}{2}}\,\max_{\omega}\int_M(|\nabla\omega|^2+A\,|\omega|^2)^2\\ \le\;& c_2\,(\lambda_l+1)^{\frac{n}{2}}\,\max_{\omega}\max_{x\in M}\left(|\nabla\omega|^2+A\,|\omega|^2\right)(x)\,\max_{\omega}\int_M(|\nabla\omega|^2+A\,|\omega|^2).\end{aligned}$$

Hence,

$$\max_{\omega}\max_{x\in M}\left(|\nabla\omega|^2+A\,|\omega|^2\right)(x)\le c_2\,(\lambda_l+1)^{\frac{n}{2}}\,\max_{\omega}\int_M(|\nabla\omega|^2+A\,|\omega|^2).$$

However,

$$\begin{aligned}&\int_M\left(|\nabla\omega|^2+A\,|\omega|^2\right)\\ =\;& -\int_M\langle\Delta\omega,\omega\rangle-\int_M\langle E(\omega),\omega\rangle+c\int_M|\omega|^2\\ \le\;& \lambda_l+K+A.\end{aligned}$$

Therefore,

$$\max_{\omega}\max_{x\in M}\left(|\nabla\omega|^2+A\,|\omega|^2\right)(x)\le c_2\,(\lambda_l+1)^{\frac{n}{2}}\,(\lambda_l+K+A).$$

The theorem is proved. □

As in section 2, we can draw the following conclusions from theorem 3.2.

Theorem 3.3. *Let (M^n,g) be a closed manifold with curvature bound $|Rm|\le K$. Let $0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_k\le\cdots$ be all the eigenvalues of the Hodge Laplacian Δ acting on the $p-$forms, and $\phi_1,\phi_2,\cdots,\phi_k,\cdots$ the corresponding orthonormal eigenforms. Then there exists a constant $c(K,d,V,n)$ such that*
(1) $|\nabla\phi_k|\le c\,(\lambda_k+1)^{\frac{n+2}{4}}$ and $|\phi_k|\le c\,(\lambda_k+1)^{\frac{n}{4}}$.
(2) For all $k>b_p$, the Betti number of the $p-$th cohomology of M,

$$\lambda_k\ge c^{-1}\,k^{\frac{2}{n}}.$$

(3) The tensor $H_p(x,y,t)$ given by

$$H_p(x,y,t)=\sum_{k=1}^{\infty}e^{-\lambda_k t}\phi_k(x)\otimes\phi_k(y)$$

is a heat kernel of Δ. Moreover,

$$|H_p(x,y,t)-\sum_{k=1}^{b(p)}\phi_k(x)\otimes\phi_k(y)|\le c\,t^{-\frac{n}{2}}$$

for all $t>0$.
(4) The following Sobolev inequality holds.

$$\left(\int_M |\omega - P(\omega)|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq c \int_M \{|d\omega|^2 + |\delta\omega|^2\}$$

for all smooth $p-$form ω on M, where $P(\omega)$ denotes the projection of ω on to the space of harmonic $p-$forms.

Proof. (1) is obvious by theorem 3.2. Using theorem 3.2, (2) follows as in the proof of (2) in theorem 2.2, where we now use a result of T. Mantuano [8] that $\lambda_{b_p+1} \geq c(n, V, d, K)$. For (3), the proof is the same as (3) in theorem 2.2. Finally, (4) follows from (3) by Theorem 1.2 in [9]. □

We also have the following corollary concerning the eigenfunctions.

Corollary 3.4. *Let (M^n, g) be a closed manifold with curvature bound $|Rm| \leq K$. Let $\phi_1, \phi_2, \cdots, \phi_k$ be orthonormal eigenfunctions of the scalar Laplacian with corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then there exists a constant $c(K, d, V, n)$ such that*

$$|\nabla d\,\phi| \leq c\,\lambda_k^{\frac{n+4}{4}},$$

where $\phi = \sum_{i=1}^k b_i\,\phi_i$ and $\sum_{i=1}^k b_i^2 = 1$.

Proof. Note that $d\phi_i$ is an eigenform for the Hodge Laplacian acting on the one forms. Now the corollary follows by applying theorem 3.2 to the one form setting with $d\phi_i$ normalized to have unit length in the L^2 sense. □

As a final remark, it is possible to make explicit of all the constants in our arguments. In particular, we could spell out their dependency on the geometric quantities d, V and K.

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